

VISCOSITY SOLUTIONS OF EIKONAL EQUATIONS ON TOPOLOGICAL NETWORKS

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ABSTRACT. In this paper we introduce a notion of viscosity solutions for Eikonal equations defined on topological networks. Existence of a solution for the Dirichlet problem is obtained via representation formulas involving a distance function associated to the Hamiltonian. A comparison theorem based on Ishii's classical argument yields the uniqueness of the solution.

1. INTRODUCTION

Several phenomena in physics, chemistry and biology, described by interaction of different media, can be translated into mathematical problems involving differential equations which are not defined on connected manifolds as usual, but instead on so-called ramified spaces. The latter can be roughly visualized as a collection of different manifolds of the same dimension (branches) with certain parts of their boundaries identified (ramification space). The simplest examples of ramified spaces are *topological networks*, which basically are graphs embedded in Euclidean space. The interaction among the collection of differential equations describing the behavior of physical quantities on the branches is described by certain transition conditions governing the interaction of the quantities across the ramification spaces.

From a mathematical point of view, the concept of ramified spaces has originally been introduced by Lumer [19] and has later been refined and specified by various authors, e. g., J. von Below and S. Nicaise [20]. Since 1980, many results have been published treating different kinds of interaction problems involving linear and quasi-linear differential equations (confer for instance Lagnese and Leugering [15], Lagnese, Leugering, and Schmidt [16], von Below and Nicaise [6]).

As far as we know, fully nonlinear equations such as Hamilton-Jacobi equations have not yet been examined to a similar extent on ramified spaces. In the present paper we attack the problem by extending the theory of viscosity solutions for Hamilton-Jacobi equations to topological networks. The major task in this context is to establish the correct transition conditions the viscosity solutions are subjected to at transition vertices. As a matter of fact, these transition conditions make up the core of our theory, as they constitute the major difference from the classical theory of viscosity solutions.

The main result of the present paper consists in the observation that the concept of viscosity solutions can indeed be appropriately extended to the class of first order Hamilton-Jacobi equations of Eikonal type on a topological network. Any generalization of existing concepts to new scenarios has to be justified by the preservation of essential

Date: January 15, 2013.

1991 Mathematics Subject Classification. Primary 49L25; Secondary 58G20, 35F20.

Key words and phrases. Hamilton-Jacobi equation; topological network; viscosity solution; comparison principle.

features. In the case of the theory of viscosity solutions, these features are uniqueness, existence, and stability. We will show that our generalization of viscosity solutions to networks will be just as “weak” to yield existence, while being sufficiently “selective” in order to ensure uniqueness and stability with respect to uniform convergence. We will also demonstrate that our definition arises as a natural selection principle, which in particular selects the distance function as the unique viscosity solution of the Dirichlet problem for the standard Eikonal equation on networks.

A different attempt to study Hamilton-Jacobi equations on networks has already been made in [1]. However, the aim of this paper deviates from the one addressed in the present paper: its main issue is to characterize the value function of controlled dynamics in \mathbb{R}^2 restricted to a network. Therefore, the choice of the Hamiltonian, which may be discontinuous with respect to the state variable, has to be restricted by assumptions ensuring both a suitable continuity property with respect to the state variable and the fact that the set of admissible controls be not empty at any point of the network. Additionally, the definition of viscosity solution characterizing the value function is different from our approach, as it involves directional derivatives of test functions in \mathbb{R}^2 along the edges. In the present paper, Hamilton-Jacobi equations and differentiation along the edges are given in an intrinsic way making use of the maps embedding the network in \mathbb{R}^N , hence the approach is intrinsically 1-dimensional. Moreover in our approach appropriate assumptions at the transition vertices guarantee the continuity of the Hamiltonian with respect to the state variable.

The existence of a viscosity solution is obtained by a representation formula involving a distance associated to the Hamiltonian (see [7], [11], [13] for corresponding results on connected domains), the solution turning out to be the maximal subsolution of the problem. Uniqueness, on the other hand, relies on a comparison principle inspired by Ishii’s classical argument for Eikonal equations [14]. In this respect, the existence of a strict subsolution plays a key role.

An important and classical problem in graph theory is the *shortest path problem*, i.e. the problem of computing in a weighted graph the distance of the vertices from a given target vertex ([4]). The weights represent the cost of running through the edges. A motivation of our work is to generalize the previous problem to the case of a running cost which varies in a continuous way along the edges. In this case the aim is to compute the distance of any point of the graph from a given target set and this in practice corresponds to solve the Eikonal equation $|Du| = \alpha(x)$ on the network with a zero-boundary condition on the target vertices. Moreover Hamilton-Jacobi equations of Eikonal type are important in several fields, for example geometric optics [5], homogenization [8, 18], singular perturbation [2], weak KAM theory [9, 10, 11], large-time behavior [12], and mean field games theory [17].

The paper is organized as follows: In section 2 we introduce the definitions of topological networks and viscosity solutions. In section 3 we collect some basic properties of viscosity solutions, in particular stability with respect to uniform convergence. Section 4 is devoted to the study of a distance function associated to the Hamilton-Jacobi equation, while section 5 presents the proof of a comparison principle. In section 6 the representation formula for the solution of the Dirichlet problem is given.

Acknowledgement 1. *This work is based on earlier results contained in the Ph.D. thesis of the first author. He would like to express his gratitude to Prof. K. P. Hadeler for the support and the guidance during the completion of the thesis.*

2. ASSUMPTIONS AND PRELIMINARY DEFINITIONS

We start with the definition of a topological network.

Definition 2.1. *Let $V = \{v_i, i \in I\}$ be a finite collection of pairwise different points in \mathbb{R}^N and let $\{\pi_j, j \in J\}$ be a finite collection of differentiable, non self-intersecting curves in \mathbb{R}^N given by*

$$\pi_j : [0, l_j] \rightarrow \mathbb{R}^N, \quad l_j > 0, \quad j \in J.$$

Set $e_j := \pi_j((0, l_j))$, $\bar{e}_j := \pi_j([0, l_j])$, and $E := \{e_j : j \in J\}$. Furthermore assume that

- i) $\pi_j(0), \pi_j(l_j) \in V$ for all $j \in J$,
- ii) $\#(\bar{e}_j \cap V) = 2$ for all $j \in J$,
- iii) $\bar{e}_j \cap \bar{e}_k \subset V$, and $\#(\bar{e}_j \cap \bar{e}_k) \leq 1$ for all $j, k \in J, j \neq k$.
- iv) *For all $v, w \in V$ there is a path with end points v and w (i.e. a sequence of edges $\{e_j\}_{j=1}^N$ such that $\#(\bar{e}_j \cap \bar{e}_{j+1}) = 1$ and $v \in \bar{e}_1, w \in \bar{e}_N$).*

Then $\bar{\Gamma} := \bigcup_{j \in J} \bar{e}_j \subset \mathbb{R}^N$ is called a (finite) topological network in \mathbb{R}^N .

If $v_i \in V \cap \bar{e}_j$ we say that e_j is incident to v_i ($e_j \text{ inc } v_i$ in short). For $i \in I$ we set $\text{Inc}_i := \{j \in J : e_j \text{ inc } v_i\}$. Observe that the parametrization of the arcs e_j induces an orientation on the edges, which can be expressed by the *signed incidence matrix* $A = \{a_{ij}\}_{i,j \in J}$ with

$$(2.1) \quad a_{ij} := \begin{cases} 1 & \text{if } v_i \in \bar{e}_j \text{ and } \pi_j(0) = v_i, \\ -1 & \text{if } v_i \in \bar{e}_j \text{ and } \pi_j(l_j) = v_i, \\ 0 & \text{otherwise.} \end{cases}$$

In the following we will study boundary value problems on $\bar{\Gamma}$. Given a nonempty set $I_B \subset I$, we define $\partial\Gamma := \{v_i, i \in I_B\}$ to be the set of *boundaries vertices*, while for $I_T := I \setminus I_B$ we call $\{v_i, i \in I_T\}$ the set of *transition vertices*. We also set $\Gamma := \bar{\Gamma} \setminus \partial\Gamma$. We always assume $i \in I_B$ whenever $\#(\text{Inc}_i) = 1$ for some $i \in I$. We remark that in applications such as the shortest path problem it is interesting to impose a boundary condition also at the internal vertices, i.e. vertices with $\#(\text{Inc}_i) > 1$.

Consider the subspace topology induced to $\bar{\Gamma}$ by \mathbb{R}^N . It coincides with the topology induced by the *path distance*

$$(2.2) \quad d(y, x) := \inf \left\{ \int_0^t |\dot{\gamma}(s)| ds : t > 0, \gamma \in B_{y,x}^t \right\} \quad \text{for } x, y \in \bar{\Gamma}, \text{ where}$$

- i) $\gamma : [0, t] \rightarrow \Gamma$ is a piecewise differentiable path in the sense that there are $t_0 := 0 < t_1 < \dots < t_{n+1} := t$ such that for any $m = 0, \dots, n$, we have $\gamma([t_m, t_{m+1}]) \subset \bar{e}_{j_m}$ for some $j_m \in J$, $\pi_{j_m}^{-1} \circ \gamma \in C^1(t_m, t_{m+1})$, and

$$\dot{\gamma}(s) = \frac{d}{ds}(\pi_{j_m}^{-1} \circ \gamma)(s).$$

- ii) $B_{y,x}^t$ is the set of all such paths with $\gamma(0) = y, \gamma(t) = x$.

For any function $u : \bar{\Gamma} \rightarrow \mathbb{R}$ and each $j \in J$ we denote by u^j the restriction of u to \bar{e}_j , i.e.

$$u^j := u \circ \pi_j : [0, l_j] \rightarrow \mathbb{R}.$$

We say that u is continuous in $\bar{\Gamma}$ and write $u \in C(\bar{\Gamma})$ if u is continuous with respect to the subspace topology of $\bar{\Gamma}$. This means that $u^j \in C([0, l_j])$ for any $j \in J$ and

$$u^j(\pi_j^{-1}(v_i)) = u^k(\pi_k^{-1}(v_i)) \quad \text{for any } i \in I, j, k \in \text{Inc}_i.$$

In a similar way we define the space of upper semicontinuous functions $\text{USC}(\bar{\Gamma})$ and the space of lower semicontinuous functions $\text{LSC}(\bar{\Gamma})$, respectively, and the space $C(\Gamma)$.

We define differentiation along an edge e_j by

$$\partial_j u(x) := \partial_j u^j(\pi_j^{-1}(x)) = \frac{\partial}{\partial x} u^j(\pi_j^{-1}(x)), \quad \text{for all } x \in e_j,$$

and at a vertex v_i by

$$\partial_j u(v_i) := \partial_j u^j(\pi_j^{-1}(v_i)) = \frac{\partial}{\partial x} u^j(\pi_j^{-1}(v_i)) \quad \text{for } j \in \text{Inc}_i.$$

A Hamiltonian $H : \bar{\Gamma} \times \mathbb{R} \rightarrow \mathbb{R}$ of *eikonal type* is a collection $(H^j)_{j \in J}$ with $H^j : [0, l_j] \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying the following conditions:

$$(2.3) \quad H^j \in C^0([0, l_j] \times \mathbb{R}), \quad j \in J,$$

$$(2.4) \quad H^j(x, p) \quad \text{is convex in } p \in \mathbb{R} \text{ for any } x \in [0, l_j], j \in J,$$

$$(2.5) \quad H^j(x, p) \rightarrow +\infty \quad \text{as } |p| \rightarrow \infty \text{ for any } x \in [0, l_j], j \in J,$$

$$(2.6) \quad H^j(\pi_j^{-1}(v_i), p) = H^k(\pi_k^{-1}(v_i), p) \quad \text{for any } p \in \mathbb{R}, i \in I, j, k \in \text{Inc}_i,$$

$$(2.7) \quad H^j(\pi_j^{-1}(v_i), p) = H^j(\pi_j^{-1}(v_i), -p) \quad \text{for any } p \in \mathbb{R}, i \in I, j \in \text{Inc}_i.$$

Remark 2.1. Assumptions (2.3)–(2.5) provide standard conditions in the theory of viscosity solutions (see [11], [13]). Assumptions (2.6)–(2.7) represent reasonable compatibility conditions of H at the vertices of $\bar{\Gamma}$, i.e. continuity at the vertices and independence of the orientation of the incident arc, respectively (the network is not oriented). A typical example of a Hamiltonian satisfying these assumptions is $H^j(x, p) := p^2 - \alpha(x)$, $j \in J$, where $\alpha(x) = \alpha^j(\pi_j^{-1}(x))$ for $x \in \bar{e}_j$ and $\alpha^j \in C^0([0, l_j])$, $\alpha^j(x) \geq 0$ for $x \in \bar{\Gamma}$, $\alpha^j(\pi_j^{-1}(v_i)) = \alpha^k(\pi_k^{-1}(v_i))$ for any $i \in I, j, k \in \text{Inc}_i$.

Definition 2.2. Let $\varphi \in C(\Gamma)$.

i) Let $x \in e_j$, $j \in J$. We say that φ is differentiable at x , if φ^j is differentiable at $\pi_j^{-1}(x)$.

ii) Let $x = v_i$, $i \in I$, $j, k \in \text{Inc}_i$, $j \neq k$. We say that φ is (j, k) -differentiable at x , if

$$(2.8) \quad a_{ij} \partial_j \varphi_j(\pi_j^{-1}(x)) + a_{ik} \partial_k \varphi_k(\pi_k^{-1}(x)) = 0,$$

where (a_{ij}) as in (2.1). Moreover, we say that φ is differentiable at x if φ is (j, k) -differentiable at x for any $j, k \in \text{Inc}_i$, $j \neq k$.

Remark 2.2. Condition (2.8) demands that the derivatives in the direction of the incident edges j and k at the vertex v_i coincide, taking into account the orientation of the edges.

On topological networks we now introduce the definition of viscosity solutions u of Hamilton-Jacobi equations of eikonal type of the form

$$(2.9) \quad H(x, Du) = 0, \quad x \in \Gamma.$$

Definition 2.3.

A function $u \in \text{USC}(\bar{\Gamma})$ is called a (viscosity) subsolution of (2.9) in Γ if the following holds:

- i) For any $x \in e_j$, $j \in J$, and for any $\varphi \in C(\Gamma)$ which is differentiable at x and for which $u - \varphi$ attains a local maximum at x , we have

$$H^j(\pi_j^{-1}(x), \partial_j \varphi_j(\pi_j^{-1}(x))) \leq 0.$$

- ii) For any $x = v_i$, $i \in I_T$, and for any φ which is (j, k) -differentiable at x and for which $u - \varphi$ attains a local maximum at x , we have

$$H^j(\pi_j^{-1}(x), \partial_j \varphi_j(\pi_j^{-1}(x))) \leq 0.$$

A function $u \in LSC(\bar{\Gamma})$ is called a (viscosity) supersolution of (2.9) in Γ if the following holds:

- i) For any $x \in e_j$, $j \in J$, and for any $\varphi \in C(\Gamma)$ which is differentiable at x and for which $u - \varphi$ attains a local minimum at x , we have

$$H^j(\pi_j^{-1}(x), \partial_j \varphi_j(\pi_j^{-1}(x))) \geq 0.$$

- ii) For any $x = v_i$, $i \in I_T$, $j \in Inc_i$, there exists $k \in Inc_i$, $k \neq j$, (which we will call i -feasible for j at x) such that for any $\varphi \in C(\Gamma)$ which is (j, k) -differentiable at x and for which $u - \varphi$ attains a local maximum at x , we have

$$H^j(\pi_j^{-1}(x), \partial_j \varphi_j(\pi_j^{-1}(x))) \geq 0.$$

A continuous function $u \in C(\Gamma)$ is called a (viscosity) solution of (2.9) if it is both a viscosity subsolution and a viscosity supersolution.

Remark 2.3. i) Let $i \in I_T$ and $\varphi \in C(\Gamma)$ be (j, k) -differentiable at x . Then by (2.6)-(2.7), we have

$$(2.10) \quad H^j(\pi_j^{-1}(x), \partial_j \varphi(\pi_j^{-1}(x))) = H^k(\pi_k^{-1}(x), \pm \partial_k \varphi(\pi_k^{-1}(x)));$$

hence in the subsolution condition, it is indifferent to require the condition for j or for k .

- ii) To simplify the notation, we set

$$H^j(x, \partial_j \varphi(x)) := H^j(\pi_j^{-1}(x), \partial_j \varphi(\pi_j^{-1}(x))).$$

Moreover, we will call $\varphi \in C(\Gamma)$ an upper (lower) (j, k) -test function of u at $x = v_i$ if it is (j, k) -differentiable at x and if $u - \varphi$ attains a local maximum (minimum) at x .

iii) It is important to observe the asymmetry in definition 2.3 regarding the subsolution and the supersolution conditions at the transition vertices. It reflects the idea that distance functions have to be solutions of (2.9) and that there is always a shortest path from a transition vertex to the boundary. In fact it is worthwhile to observe that if supersolutions were defined similarly to subsolutions, the conditions in general would not be satisfied by $\inf\{d(y, x) : y \in \partial\Gamma\}$, which is, as we will see in section 6, the solution of $|Du|^2 - 1 = 0$ with zero boundary conditions.

iv) Taking (2.10) into account, it is easily seen that a viscosity solution u of (2.9) satisfies the equation in a pointwise sense at any point $x \in \Gamma$ where it is differentiable.

As the conditions in definition 2.3 are of pointwise character, they can also be imposed on subsets of Γ . Hence, for $\Omega \subseteq \Gamma$ we denote by $\mathcal{S}(\Omega)$ ($\mathcal{S}^+(\Omega)$ or $\mathcal{S}^-(\Omega)$, respectively) the space of solutions (supersolutions or subsolutions, respectively) of (2.9) in Ω . For short, we set $\mathcal{S} := \mathcal{S}(\Gamma)$, $\mathcal{S}^+ := \mathcal{S}^+(\Gamma)$, and $\mathcal{S}^- := \mathcal{S}^-(\Gamma)$.

3. SOME BASIC PROPERTIES OF VISCOSITY SOLUTIONS

In this section we discuss some basic properties of the viscosity solutions introduced in the previous section.

Proposition 3.1. *Let u, v be subsolutions (supersolutions) of (2.9) in Γ . Then $w := \max\{u, v\}$ ($w := \min\{u, v\}$) is a subsolution (a supersolution) of (2.9) in Γ .*

Proof. We only consider the case $x = v_i$, $i \in I$, as otherwise the argument is standard. Let $u, v \in \text{USC}(\Gamma)$ be two subsolutions at x and observe that $w = \max\{u, v\} \in \text{USC}(\Gamma)$. Let $j, k \in \text{Inc}_i$ and let φ be an upper (j, k) -test function of w at x . If $w(x) = u(x)$ (similarly in the other case) then φ is an upper (j, k) -test function of u at x , implying

$$H^j(x, \partial_j \varphi(x)) \leq 0.$$

Hence w is a subsolution.

Let $u, v \in \text{LSC}(\Gamma)$ be two supersolutions at x , whence $w = \min\{u, v\} \in \text{LSC}(\Gamma)$. Assume $w(x) = u(x)$ (similarly in the other case). Hence for any $j \in \text{Inc}_i$ there exists $k \in \text{Inc}_i$, $k \neq j$, such that for any lower (j, k) -test function φ of u at x we have

$$H^j(x, \partial_j \varphi(x)) \geq 0.$$

Hence k is i -feasible for j also with respect to w . \square

Proposition 3.2. *Assume $H_n(x, p) \rightarrow H(x, p)$ uniformly for $n \rightarrow \infty$ (i.e. $H_n^j(\pi_j^{-1}(x), p) \rightarrow H^j(\pi_j^{-1}(x), p)$ uniformly for $(x, p) \in \bar{e}_j \times \mathbb{R}$ for any $j \in J$). For any $n \in \mathbb{N}$ let u_n be a solution of*

$$(3.1) \quad H_n(x, Du) = 0, \quad x \in \Gamma,$$

and assume $u_n \rightarrow u$ uniformly in Γ for $n \rightarrow \infty$. Then u is a solution of (2.9).

Proof. We treat the case $x = v_i$, $i \in I_T$, as otherwise the argument is standard (see [3]). We first prove that u is a *subsolution*. Choose any $j, k \in \text{Inc}_i$, $j \neq k$, along with an upper (j, k) -test function φ of u at x . Consider the auxiliary function $\varphi_\delta(y) := \varphi(y) + \delta d(x, y)^2$ for $\delta > 0$. Observe that $\partial_m(d(x, \cdot)^2)(\pi_m^{-1}(x)) = 0$ for $m = j$ and $m = k$, hence $d(x, \cdot)^2$ is (j, k) differentiable at x . Then φ_δ is an upper (j, k) -test function of u at x and there exists $r > 0$ such that $u - \varphi_\delta$ attains a strict local maximum w.r.t. $\bar{B}_r(x)$ at x , where $B_r(x) := \{y \in \Gamma : d(x, y) < r\}$. Observe that x is a strict maximum point for $u - \varphi_\delta$ also in $\bar{B} := \bar{B}_r(x) \cap (\bar{e}_j \cup \bar{e}_k)$. Now choose a sequence $\omega_n \rightarrow 0$ for $n \rightarrow \infty$ with

$$(3.2) \quad \sup_{\Gamma} |u(x) - u_n(x)| \leq \omega_n$$

and let y_n be a maximum point for $u_n - \varphi_\delta$ in \bar{B} . Up to a subsequence, $y_n \rightarrow z \in \bar{B}$. Moreover,

$$u(x) - \varphi_\delta(x) - \omega_n \leq u_n(x) - \varphi_\delta(x) \leq u_n(y_n) - \varphi_\delta(y_n) \leq u(y_n) - \varphi_\delta(y_n) + \omega_n.$$

For $n \rightarrow \infty$, we get $u(x) - \varphi_\delta(x) \leq u(z) - \varphi_\delta(z)$. As x is a strict maximum point, we conclude $x = z$. Invoking

$$u(x) + \varphi_\delta(y_n) - \varphi_\delta(x) - \omega_n \leq u_n(y_n) \leq u(y_n) + \omega_n$$

we altogether get

$$(3.3) \quad \lim_{n \rightarrow \infty} y_n = x, \quad \lim_{n \rightarrow \infty} u_n(y_n) = u(x)$$

We distinguish two cases:

Case 1: $y_n \neq x$. Then $y_n \in e_m$ with either $m = j$ or $m = k$. Since $u_n - \varphi_\delta$ attains a maximum at y_n and as φ_δ is differentiable at y_n with $\partial_m \varphi_\delta(y_n) = \partial_m \varphi(y_n) + 2\delta a_{im} d(x, y_n)$, we have

$$(3.4) \quad H_n^m(y_n, \partial_m \varphi(y_n) + 2\delta a_{im} d(x, y_n)) \leq 0.$$

with either $m = j$ or $m = k$.

Case 2: $y_n = x$. Then $\partial_m \varphi_\delta(y_n) = \partial_m \varphi(y_n)$ for $m = j$ and $m = k$ and therefore

$$(3.5) \quad H_n^j(y_n, \partial_j \varphi_\delta(y_n)) \leq 0.$$

By (3.3), (3.4), (3.5) and recalling (2.10) we get for $n \rightarrow \infty$

$$H^j(x, \partial_j \varphi(x)) \leq 0.$$

To show that u is a *supersolution*, we assume by contradiction that there exists $j \in \text{Inc}_i$ such that for any $k \in \text{Inc}_i$, $k \neq j$, there exists a lower (j, k) -test function φ_k of u at x for which

$$(3.6) \quad H^j(x, \partial_j \varphi_k(x)) < 0.$$

By adding a quadratic function of the form $-\alpha_k d(x, y)^2$ to the function φ_k we may assume that there exists $r > 0$ such that $u - \varphi_k$ attains a strict minimum in $\bar{B}_r(x)$ at x . Observe that x is a strict minimum point of $u - \varphi_k$ also in $\bar{B}_k := \bar{B}_r(x) \cap (\bar{e}_j \cup \bar{e}_k)$. Now for any $n \in \mathbb{N}$ there exists $k_n \in \text{Inc}_i$, $k_n \neq j$, which is i -feasible for j with respect to u_n . Up to a subsequence, we may assume that there exists $k \in \text{Inc}_i$ such that $k_n = k$ for any n .

Let y_n be a minimum point of $u_n - \varphi_k$ in \bar{B}_k and let ω_n be as in (3.2). Similarly to the subsolution case, we can prove that (3.3) holds. If $y_n \neq x$, we obtain

$$(3.7) \quad H_n^m(y_n, \partial_j \varphi_k(y_n)) \geq 0$$

for either $m = j$ or $m = k$. If $y_n = x$, we get

$$(3.8) \quad H_n^j(y_n, \partial_j \varphi_k(y_n)) \geq 0.$$

Hence by (3.3), (3.7), and (3.8), we get for $n \rightarrow \infty$

$$H^j(x, \partial_j \varphi_k(x)) \geq 0,$$

which is a contradiction to (3.6). \square

The proof of the following proposition is given in [3, Prop.II.4.1], for example.

Proposition 3.3. *Let K be a compact subset of Γ and let $u \in \mathcal{S}^-(K)$. Then there exists a constant C_K depending only on K such that*

$$(3.9) \quad |u(x) - u(y)| \leq C_K d(x, y).$$

The proof of the next two propositions is very similar to the one of Prop.3.2.

Proposition 3.4. *Let $\mathcal{T} \subset \mathcal{S}^-$ ($\mathcal{T} \subset \mathcal{S}^+$) and set $u(x) := \sup\{v(x) | v \in \mathcal{T}\}$ ($u(x) := \inf\{v(x) | v \in \mathcal{T}\}$) for $x \in \Gamma$. Suppose that $u \in C(\Gamma)$. Then $u \in \mathcal{S}^-$ ($u \in \mathcal{S}^+$).*

Proof. To prove that $u(x) = \sup\{v(x) \mid v \in \mathcal{T}\}$ is a subsolution, we only consider the case $x = v_i$, $i \in I_T$. Consider $j, k \in Inc_i$, $j \neq k$, and an upper (j, k) -test function φ of u at x . Set $\varphi_\delta(y) := \varphi(y) + \delta d(x, y)^2$ for $\delta > 0$. Then φ_δ is an upper (j, k) -test function of u at x and there exists $r > 0$ such that $u - \varphi_\delta$ has a strict local maximum point in $\bar{B}_r(x)$ at x . Observe that x is a strict maximum point for $u - \varphi_\delta$ also in $\bar{B} := \bar{B}_r(x) \cap (\bar{e}_j \cup \bar{e}_k)$. Let $u_n \in \mathcal{S}$ be such that

$$u(x) - u_n(x) \leq \frac{1}{n}$$

and let y_n be a maximum point for $u_n - \varphi_\delta$ in \bar{B} . Up to a subsequence, $y_n \rightarrow z \in \bar{B}$. Moreover,

$$u(x) - \varphi_\delta(x) - \frac{1}{n} \leq u_n(x) - \varphi_\delta(x) \leq u_n(y_n) - \varphi_\delta(y_n) \leq u(y_n) - \varphi_\delta(y_n).$$

For $n \rightarrow \infty$, we obtain $u(x) - \varphi_\delta(x) \leq u(z) - \varphi_\delta(z)$, implying $x = z$, as x is a strict maximum point. Moreover, by

$$u(x) + \varphi_\delta(y_n) - \varphi_\delta(x) \leq u_n(y_n) \leq u(y_n)$$

we get

$$\lim_{n \rightarrow \infty} y_n = x, \quad \lim_{n \rightarrow \infty} u_n(y_n) = u(x),$$

and we conclude as in Proposition 3.2.

Similarly to the proof of Proposition 3.2 one can also show that $u(x) := \inf\{v(x) \mid v \in \mathcal{T}\}$ is a supersolution. \square

Proposition 3.5. *Let $\mathcal{T} \subset \mathcal{S}$ and let $u(x) := \inf\{v(x) \mid v \in \mathcal{T}\}$ for $x \in \Gamma$. Assume that $u(x) \in \mathbb{R}$ for some $x \in \Gamma$. Then $u \in \mathcal{S}$.*

Proof. By (3.9) all $v \in \mathcal{S}$ are uniformly Lipschitz continuous. As $u(x) \in \mathbb{R}$, we thus have $u(y) \in \mathbb{R}$ for any $y \in \Gamma$. Moreover, u is Lipschitz continuous on Γ . Next observe that by Proposition 3.4 u is a supersolution of (2.9).

In order to prove that u is also a subsolution we once more invoke (and only sketch) the argument used in Proposition 3.2. Consider $x = v_i$, $i \in I_T$, $j, k \in Inc_i$, $j \neq k$, and an upper (j, k) -test function φ of u at x . Define the auxiliary function $\varphi_\delta(y) := \varphi(y) + \delta d(x, y)^2$ for $\delta > 0$. Then φ_δ is an upper (j, k) -test function of u at x and there exists $r > 0$ such that $u - \varphi_\delta$ has a strict local maximum point in $\bar{B}_r(x)$ at x . Observe that x is a strict maximum point for $u - \varphi_\delta$ also in $\bar{B} := \bar{B}_r(x) \cap (\bar{e}_j \cup \bar{e}_k)$. Let $u_n \in \mathcal{T}$ be such that

$$u(x) - u_n(x) \geq -\frac{1}{n}$$

and let y_n be a maximum point for $u_n - \varphi_\delta$ in \bar{B} . Up to a subsequence, $y_n \rightarrow z \in \bar{B}$. Moreover,

$$u(x) - \varphi_\delta(x) \leq u_n(x) - \varphi_\delta(x) \leq u_n(y_n) - \varphi_\delta(y_n) \leq u(y_n) - \varphi_\delta(y_n) + \frac{1}{n}.$$

Hence we obtain (3.3). Arguing as in Proposition 3.2 we conclude $H^j(x, \partial_j \varphi(x)) \leq 0$. \square

4. A DISTANCE FUNCTION FOR HAMILTON-JACOBI EQUATIONS

In this section we assume

$$(4.1) \quad \mathcal{S}^-(\Gamma) \neq \emptyset,$$

i.e. there exists a subsolution of (2.9) in Γ . We introduce a distance function related to the Hamiltonian H on the network. For $x, y \in \Gamma$ define

$$(4.2) \quad S(y, x) = \inf \left\{ \int_0^t L(\gamma(s), \dot{\gamma}(s)) ds : t > 0, \gamma \in B_{y,x}^t \right\},$$

where $B_{y,x}^t$ as in (2.2) and

$$L^j(x, q) := \sup_{p \in \mathbb{R}} \{p q - H^j(x, p)\} = \sup_{p \in \mathbb{R}} \{p q - H^j(\pi_j^{-1}(x), p)\}$$

for any $j \in J$, $x \in \bar{e}_j$. Note that the distance defined by (4.2) coincides with the distance defined by (2.2) for $H(x, p) = |p|^2 - 1$. The next proposition summarizes some properties of S .

Proposition 4.1. *S is a Lipschitz continuous distance on $\Gamma \times \Gamma$. Moreover,*

- i) *for any $y \in \Gamma$ we have $S(y, \cdot) \in \mathcal{S}^-(\Gamma) \cap \mathcal{S}(\Gamma \setminus \{y\})$,*
- ii) *for any $x, y \in \Gamma$ we have*

$$(4.3) \quad S(y, x) = \max\{u(x) \mid u \in \mathcal{S}^-(\Gamma) \text{ s.t. } u(y) = 0\}.$$

Proof. By (3.9) any subsolution u of (2.9) in Γ is Lipschitz continuous. Integrating along a path joining x and y we get

$$(4.4) \quad u(x) - u(y) \leq S(y, x) \quad \text{for any } x, y \in \Gamma.$$

Thus by (4.1) $S(y, x)$ is finite for any $x, y \in \bar{\Gamma}$.

By the coercitivity of H assumed in (2.5) there exists constant $R > 0$, M such that $L(x, q) \leq M$ (i.e. $L^j(\pi_j^{-1}(x), q) \leq M$) for any $x \in \Gamma$, $q \in B(0, R)$, and therefore $S(y, x) \leq C_R d(y, x)$ (see [12, Prop.5.1] for details). Moreover, given $x, y, z \in \Gamma$, the juxtaposition of two curves in $B_{y,z}$ and $B_{z,x}$ gives a curve in $B_{y,x}$, whence

$$S(y, x) \leq S(y, z) + S(z, x).$$

$S(y, \cdot)$ is a subsolution: In order to prove that $S(y, \cdot)$ is a subsolution at $x_0 \in \Gamma$ we distinguish two cases:

Case 1: $x_0 \notin \{v_i, i \in I_T\}$. Assume $x_0 \in e_j$ for some $j \in J$ and let ψ be an upper test function of $S(y, \cdot)$ at x_0 . It follows $S(y, x_0) - \psi_j(\pi_j^{-1}(x_0)) \geq S(y, x) - \psi_j(\pi_j^{-1}(x))$ for $x \in B_r(x_0) \cap e_j$. Set $t_0 := \pi_j^{-1}(x_0)$, fix $q \in \mathbb{R}$, and choose h sufficiently small in such a way that $t_0 - hq \in (0, l_j)$. Define the curve $\gamma_h : [0, h] \rightarrow \Gamma$ by $\gamma_h(s) := \pi_j(\frac{s}{h}t_0 + (1 - \frac{s}{h})t_h)$, where $t_h := t_0 - hq$ and set $x_{hq} := \pi_j(t_0 - hq)$. Hence

$$\begin{aligned} \partial \psi(x_0) q &= \partial_j \psi_j(t_0) q = \lim_{h \rightarrow 0^+} \frac{\psi_j(t_0) - \psi_j(t_h)}{h} \leq \lim_{h \rightarrow 0^+} \frac{S(y, x_0) - S(y, x_{hq})}{h} \\ &\leq \lim_{h \rightarrow 0^+} \frac{S(x_{hq}, x_0)}{h} \leq \lim_{h \rightarrow 0^+} \frac{1}{h} \int_0^h L(\gamma_h(s), \dot{\gamma}_h(s)) ds = \\ &\lim_{h \rightarrow 0^+} \frac{1}{h} \int_0^h L^j\left(\frac{s}{h}t_0 + \left(1 - \frac{s}{h}\right)t_h, q\right) ds = L^j(t_0, q) = L(x_0, q). \end{aligned}$$

Hence $H(x, \partial\psi(x_0)) = \sup_q \{\partial\psi(x_0)q - L(x_0, q)\} \leq 0$.

Case 2: $x_0 \in \{v_i, i \in I_T\}$. Assume $x_0 = v_i$ and let ψ be an upper (j, k) -test function of $S(y, \cdot)$ at x_0 . Set $t_m := \pi_m^{-1}(x_0)$, $m = j, k$, and observe that for any $q \in \mathbb{R}$ and for h sufficiently small we have $t_m - h(-a_{im}q) \in (0, l_m)$ (or equivalently $\pi_m(t_m - h(-a_{im}q)) \in e_m$) for $m = j, k$. Arguing as in case 1, we get

$$(4.5) \quad \begin{aligned} \partial_j \psi_j(\pi_j^{-1}(x_0))(-a_{ij}q) &\leq L^j(\pi_j^{-1}(x_0), -a_{ij}q), \\ \partial_k \psi_k(\pi_k^{-1}(x_0))(-a_{ik}q) &\leq L^k(\pi_k^{-1}(x_0), -a_{ik}q). \end{aligned}$$

Moreover, since $a_{ij}\partial_j\psi(\pi_j^{-1}(x_0)) + a_{ik}\partial_k\psi_k(\pi_k^{-1}(x_0)) = 0$ and $L^j(\pi_j^{-1}(x_0), q) = L^k(\pi_k^{-1}(x_0), q)$, we get

$$(4.6) \quad \begin{aligned} \partial_j \psi_j(\pi_j^{-1}(x_0))(a_{ij}q) &\leq L^j(\pi_j^{-1}(x_0), a_{ij}q), \\ \partial_k \psi_k(\pi_k^{-1}(x_0))(a_{ik}q) &\leq L^k(\pi_k^{-1}(x_0), a_{ik}q). \end{aligned}$$

By (4.5) and (4.6) it follows $\partial_m \psi_m(\pi_m^{-1}(x_0))q \leq L^m(\pi_m^{-1}(x_0), q)$ for $m = j, k$, whence $H(x, \partial\psi(x_0)) = \sup_q \{\partial\psi(x_0)q - L(x_0, q)\} \leq 0$. By (4.4) and since $S(y, \cdot)$ is a subsolution with $S(y, y) = 0$, we finally obtain (4.3).

S(y, \cdot) is a supersolution in $\Gamma \setminus \{y\}$: In order to prove that $S(y, \cdot)$ is a supersolution at $x \neq y$ we only consider the case $x = v_i$, $i \in I_T$, being the other case standard. Assume that $u(\cdot) = S(y, \cdot)$ is not a supersolution at x . By definition there exists an index $j \in Inc_i$ for which there does not exist any i -feasible index $k \in Inc_i$, $k \neq j$. Hence for any $k \in K = Inc_i \setminus \{j\}$ there exists a lower (j, k) -test function φ_k of u at x with

$$(4.7) \quad H^j(x, \partial_j \varphi_k(x)) < 0.$$

By Remark 2.3(i) we have

$$(4.8) \quad H^j(x, \partial_j \varphi_k(x)) = H^k(x, \partial_k \varphi_k(x)) < 0.$$

It is not restrictive to assume that $u(x) = \varphi_k(x)$ for any $k \in K$. By adding a term of the form $-\alpha d(z, x)^2$ we may assume that $u - \varphi_k$ attains a strict minimum point at x . Hence by (2.3) and (4.8) there exists $r > 0$ such that for all $k \in K$ and $z \in B_r(x) \setminus \{x\}$

$$(4.9) \quad \begin{aligned} u(z) - \varphi_k(z) &> 0 \\ H^j(z, \partial_j \varphi_k(z)) &< 0, \quad H^k(z, \partial_k \varphi_k(z)) < 0. \end{aligned}$$

Let $\xi > 0$ be such that

$$(4.10) \quad u(z) - \varphi_k(z) > \xi \quad \text{for all } k \in K \text{ and } z \in \partial B_r(x).$$

Define $\tilde{\varphi}_k(z) := \varphi_k(z) + \xi$ and $\tilde{v} : \{x\} \cup \bigcup_{k \in Inc_i} \bar{e}_k \rightarrow \mathbb{R}$ by

$$\tilde{v}(z) := \begin{cases} \max_{k \in K} \tilde{\varphi}_k(z), & \text{if } z \in \bar{e}_j, \\ \tilde{\varphi}_k(z), & \text{if } z \in \bar{e}_k, k \in K. \end{cases}$$

We claim that \tilde{v} is a subsolution of (2.9) in $B_r(x)$.

Case 1: Consider $z \in B_r(x) \cap e_l$. If $l \in K$, then $\tilde{v}(z) = \tilde{\varphi}_l(z)$ and the claim follows by (4.9). If $l = j$, by (4.9) we have

$$H^k(z, \partial_k \tilde{\varphi}_k(z)) < 0 \quad \forall k \in K$$

and the subsolution condition follows by Proposition 3.1.

Case 2: Consider $z = x$. First assume $l, m \in K$, $l \neq m$, and let ψ be an upper (l, m) -test function of \tilde{v} at x . Set

$$d_l := a_{il}\partial_l\psi(x), d_m := a_{im}\partial_m\psi(x), \eta_l := a_{il}\partial_l\tilde{\varphi}_l(x), \eta_m := a_{im}\partial_m\tilde{\varphi}_m(x).$$

As ψ is (l, m) -differentiable at x , we have $d_l + d_m = 0$. If $d_l \leq 0$, we have $d_l \geq \eta_l$ by the definition of \tilde{v} and by the fact that $\tilde{v} - \psi$ attains a local maximum at $z = x$. Hence $|d_l| \leq |\eta_l|$. Similarly, if $d_m \leq 0$, we have $d_m \geq \eta_m$, implying $|d_m| \leq |\eta_m|$. We therefore conclude that

$$|\partial_l\psi(x)| = |\partial_m\psi(x)| \leq \max\{|\partial_l\tilde{\varphi}_l(x)|, |\partial_m\tilde{\varphi}_m(x)|\}.$$

By the assumptions on H (see (2.3)-(2.7)) the function $h : \mathbb{R} \rightarrow \mathbb{R}$, $p \mapsto H^s(x, p)$ for $s \in \text{Inc}_i$ is independent of s , symmetric at $p = 0$, and strictly increasing in $|p|$. Hence by (4.8)

$$\begin{aligned} H^l(x, \partial_l\psi(x)) &= h(\partial_l\psi(x)) \leq \max\{h(\partial_l\tilde{\varphi}_l(x)), h(\partial_m\tilde{\varphi}_m(x))\} = \\ &= \max\{H^l(x, \partial_l\tilde{\varphi}_l(x)), H^m(x, \partial_m\tilde{\varphi}_m(x))\} < 0. \end{aligned}$$

Assume now that ψ is an upper (j, l) -test function of \tilde{v} at x and set

$$d_j := a_{ij}\partial_j\psi(x), d_l := a_{il}\partial_l\psi(x), e_j := \max_{k \in K} a_{ij}\partial_j\tilde{\varphi}_k(x), \eta_l := a_{il}\partial_l\tilde{\varphi}_l(x).$$

As above, we have $|d_j| \leq |e_j|$ if $d_j \leq 0$ and $|d_m| \leq |\eta_m|$ if $d_m \leq 0$. Hence we get

$$|\partial_j\psi(x)| = |\partial_l\psi(x)| \leq \max\{\max_{k \in K} |\partial_k\tilde{\varphi}_k(x)|, |\partial_l\tilde{\varphi}_l(x)|\},$$

and therefore by (4.8)

$$\begin{aligned} H^j(x, \partial_j\psi(x)) &= h(\partial_j\psi(x)) \leq \max\{\max_{k \in K} h(\partial_k\tilde{\varphi}_k(x)), h(\partial_l\tilde{\varphi}_l(x))\} = \\ &= \max\{\max_{k \in K} H^k(x, \partial_k\tilde{\varphi}_k(x)), H^l(x, \partial_l\tilde{\varphi}_l(x))\} < 0. \end{aligned}$$

Hence \tilde{v} is a viscosity subsolution in $B_r(y)$. Define the function $v : \Gamma \rightarrow \mathbb{R}$ by

$$v(z) := \begin{cases} \max\{\tilde{v}(z), u(z)\}, & \text{if } z \in B_t(x), \\ u(z), & \text{if } z \in \Gamma \setminus B_t(x). \end{cases}$$

By (4.10), v is continuous, $v = u$ outside $B_r(x)$, and v is a subsolution of (2.9) in Γ . Since $v(x) = \tilde{v}(x) > u(x)$, we get a contradiction to (4.3). \square

5. A COMPARISON THEOREM

This section is devoted to the proof of a comparison theorem for problem (2.9).

Theorem 5.1. *Assume that there exists a closed subset $K \subset \Gamma$ and a function $f \in C(\Gamma)$ with $f(x) < 0$ for all $x \in \Gamma \setminus K$. Moreover, let u be a subsolution of*

$$(5.1) \quad H(x, Du) = f(x), \quad x \in \Gamma \setminus K,$$

and let v be a supersolution of (2.9) in $\Gamma \setminus K$. If $u \leq v$ on $\partial\Gamma \cup K$, then $u \leq v$ in $\bar{\Gamma}$.

Proof. Assume by contradiction that there exists $z \in \Gamma \setminus K$ such that

$$(5.2) \quad u(z) - v(z) = \max_{\bar{\Gamma}} \{u - v\} = \delta > 0.$$

For $\epsilon > 0$ define $\Phi_\epsilon : \bar{\Gamma} \times \bar{\Gamma} \rightarrow \mathbb{R}$ by

$$\Phi_\epsilon(x, y) := u(x) - v(y) - \epsilon^{-1}d(x, y)^2.$$

As Φ_ϵ is upper semicontinuous there exists a maximum point (p_ϵ, q_ϵ) for Φ_ϵ in $\bar{\Gamma}^2$. By $\Phi_\epsilon(z, z) \leq \Phi_\epsilon(p_\epsilon, q_\epsilon)$ we get

$$(5.3) \quad \epsilon^{-1}d(p_\epsilon, q_\epsilon)^2 \leq u(p_\epsilon) - v(q_\epsilon) - \delta,$$

whence

$$(5.4) \quad \lim_{\epsilon \rightarrow 0} d(p_\epsilon, q_\epsilon) = 0.$$

By the compactness of $\bar{\Gamma}$, there exists $\bar{p} \in \bar{\Gamma}$ such that $p_\epsilon, q_\epsilon \rightarrow \bar{p}$. By (5.3) and the Lipschitz continuity of u (see (3.9)) we get

$$\epsilon^{-1}d(p_\epsilon, q_\epsilon)^2 \leq u(p_\epsilon) - u(q_\epsilon) + u(q_\epsilon) - v(q_\epsilon) - \delta \leq Ld(p_\epsilon, q_\epsilon)$$

and therefore

$$(5.5) \quad \lim_{\epsilon \rightarrow 0^+} \epsilon^{-1}d(p_\epsilon, q_\epsilon) = 0.$$

Moreover, by (5.3)-(5.4) we have $\bar{p} \in \Gamma \setminus K$ as well as $p_\epsilon, q_\epsilon \in \Gamma \setminus K$ for a sufficiently small choice of ϵ . Next observe that it is possible to assume that there exists a unique path $\gamma = \gamma_\epsilon$ of length $d(p_\epsilon, q_\epsilon)$ in Γ connecting p_ϵ and q_ϵ which runs through at most one vertex v_i , $i \in I$. We distinguish several cases (for simplicity we set $p := p_\epsilon$, $q := q_\epsilon$).

Case 1: There are indices $i \in I$ and $j, k \in \text{Inc}_i$ such that $p \in e_j$, $q \in e_k$ and such that γ runs through v_i . We observe that the functions $\varphi_p(x) := \epsilon^{-1}d(p, x)^2$ and $\varphi_q(x) := \epsilon^{-1}d(x, q)^2$ are differentiable at q and p , respectively. In fact, if

$$(5.6) \quad \tilde{p} = \pi_j^{-1}(p), \quad \tilde{q} = \pi_k^{-1}(q),$$

we have

$$\partial_j \varphi_q^j(\tilde{p}) = \epsilon^{-1}d(p, q)a_{ij}, \quad \partial_k \varphi_p^k(\tilde{q}) = \epsilon^{-1}d(p, q)a_{ik}.$$

Observe that $u - \varphi_q$ has a maximum point at p and $v + \varphi_p$ has a minimum point at q , whence

$$H^j(p, \partial_j \varphi_q(p)) = H^j(\tilde{p}, \epsilon^{-1}d(p, q)a_{ij}) \leq f(p)$$

$$H^k(q, \partial_k \varphi_p(q)) = H^k(\tilde{q}, -\epsilon^{-1}d(p, q)a_{ik}) \geq 0.$$

We denote by ω_m , $m = j, k$, the modulus of continuity of H^m with respect to $(x, p) \in \bar{e}_m \times \mathbb{R}$. By (2.6) and (2.7) there is some $\eta > 0$ such that for sufficiently small $\epsilon > 0$ we have

$$\begin{aligned} \eta &\leq -f(p) \leq H^k(\tilde{q}, -\epsilon^{-1}d(p, q)a_{ik}) - H^j(\tilde{p}, \epsilon^{-1}d(p, q)a_{ij}) \leq \\ &H^k(v_i, -\epsilon^{-1}d(p, q)a_{ik}) - H^j(v_i, \epsilon^{-1}d(p, q)a_{ij}) + \omega_k(d(v_i, q)) + \omega_j(d(v_i, p)) = \\ &H^j(v_i, \epsilon^{-1}d(p, q)a_{ik}) - H^j(v_i, \epsilon^{-1}d(p, q)a_{ij}) + \omega_k(d(v_i, q)) + \omega_j(d(v_i, p)) \\ &\leq \omega_j(\epsilon^{-1}d(p, q)(|a_{ij}| + |a_{ik}|)) + \omega_k(d(v_i, q)) + \omega_j(d(v_i, p)). \end{aligned}$$

By (5.5) we get a contradiction for $\epsilon \rightarrow 0$.

Case 2: There are indices $i \in I$ and $j \in Inc_i$ such that $p \in e_j$ and $q = v_i$. As $q \in \Gamma$, we have $i \in I_T$. Setting φ_p and φ_q as above and using a notation similar to (5.6) we have

$$(5.7) \quad \partial_j \varphi_p^j(\tilde{q}) = -\epsilon^{-1} d(p, q) a_{ij}, \quad \partial_k \varphi_p^k(\tilde{q}) = \epsilon^{-1} d(p, q) a_{ik} \quad \text{for all } k \in Inc_i, k \neq j.$$

Hence

$$(5.8) \quad a_{ij} \partial_j \varphi_p^j(\tilde{q}) + a_{ik} \partial_k \varphi_p^k(\tilde{q}) = (-a_{ij}^2 + a_{ik}^2) \epsilon^{-1} d(p, q) = 0 \quad \text{for all } k \in Inc_i, k \neq j.$$

Thus φ_p is (j, k) -differentiable at q for all $k \in Inc_i, k \neq j$. Moreover,

$$\partial_j \varphi_q^j(\tilde{p}) = \epsilon^{-1} d(p, q) a_{ij}.$$

Since $u - \varphi_q$ has a maximum point at p , it follows

$$(5.9) \quad H^j(p, \partial_j \varphi_q(p)) = H^j(\tilde{p}, \epsilon^{-1} d(p, q) a_{ij}) \leq f(p).$$

Moreover, since $v + \varphi_p$ has a minimum point at $q = v_i$, there is an i -feasible index $k_0 \in Inc_i, k_0 \neq j$, for j . By (5.8), φ_p is (j, k_0) -differentiable, whence we obtain

$$(5.10) \quad H^j(q, \partial_j \varphi_p(q)) = H^j(\tilde{q}, -\epsilon^{-1} d(p, q) a_{ik}) \geq 0.$$

Subtracting (5.9) from (5.10) we derive a contradiction as in case 1.

Case 3: There are indices $i \in I$ and $j \in Inc_i$ such that $p = v_i$ and $q \in e_j$. We proceed as in case 2, observing that the definition of subsolutions is less restrictive than the definition of supersolutions and therefore no extra argument is required.

Case 4: There are indices $j \in J$ such that $p, q \in e_j$ and $p \neq q$. Setting φ_p and φ_q as above and using a notation similar to (5.6) we have

$$(5.11) \quad \partial_j \varphi_q^j(p) = -\partial_j \varphi_p^j(q).$$

Hence we have

$$\begin{aligned} H^j(p, \partial_j \varphi_q(p)) &\leq f(p), \\ H^j(q, -\partial_j \varphi_p(q)) &= H^j(q, \partial_j \varphi_q(p)) \geq 0 \end{aligned}$$

and we conclude as in the previous cases.

Case 5: We finally assume that $p = q$. Assume $p = q = v_i$ for $i \in I_T$ (the case $p, q \in e_j$ for $j \in J$ is similar). Then

$$\partial_j \varphi_q^j(\pi_j^{-1}(v_i)) = \partial_j \varphi_p^j(\pi_j^{-1}(v_i)) = 0$$

for all $j \in Inc_i$. In particular for each choice of $j, k \in Inc_i$, both φ_q and $-\varphi_p$ are (j, k) -differentiable and we get a contradiction as in the previous cases. \square

6. REPRESENTATION FORMULA FOR VISCOSITY SOLUTIONS

In this section we give a representation formula for the solution of the Dirichlet problem

$$(6.1) \quad H(x, Du) = 0, \quad x \in \Gamma,$$

$$(6.2) \quad u = g, \quad x \in \partial\Gamma.$$

In addition to (2.3)-(2.7) and (4.1), in this section we assume that

there exist a closed (possibly empty) subset $K \subset \Gamma$, a differentiable function ψ , and $h \in C(\Gamma)$ with $h < 0$ in $\Gamma \setminus K$ such that

$$H(x, D\psi) \leq h(x), \quad x \in \Gamma \setminus K,$$

i.e. ψ is differentiable the sense of Definition 2.2 and a strict subsolution in $\Gamma \setminus K$.

Proposition 6.1. *Let $g : \bar{\Gamma} \rightarrow \mathbb{R}$ be a continuous function satisfying*

$$(6.4) \quad g(x) - g(y) \leq S(y, x) \quad \text{for any } x, y \in K \cup \partial\Gamma,$$

where S is the distance defined in (4.2). Then the unique viscosity solution of (6.1)–(6.2) is given by

$$u(x) := \min\{g(y) + S(y, x) : y \in K \cup \partial\Gamma\}.$$

Proof. By Proposition 3.5 u is a solution of (2.9). Observe that we have $u(x) \neq g(x)$ for $x \in K \cup \partial\Gamma$ if and only if there is some $z \in K \cup \partial\Gamma$ such that $g(x) > S(z, x) + g(z)$. However, this is ruled out by assumption (6.4). Hence u is a solution of (6.1)–(6.2).

Assume that there exists another solution v of (6.1)–(6.2). For $\theta \in (0, 1)$ define $u_\theta := \theta u + (1 - \theta)\psi$, where ψ as in (6.3). By adding a constant it is not restrictive to assume that ψ is sufficiently small in such a way that

$$(6.5) \quad u_\theta(x) \leq u(x), \quad x \in \bar{\Gamma}.$$

First, let $x \in e_j \cap (\Gamma \setminus K)$ for some $j \in J$ and let φ be an upper test function of u at x . Setting $\varphi_\theta := \theta\varphi + (1 - \theta)\psi$ we obtain by means of convexity

$$(6.6) \quad H^j(x, \partial_j \varphi_\theta) \leq \theta H^j(x, \partial_j \varphi) + (1 - \theta) H^j(x, \partial_j \psi) \leq (1 - \theta) h^j(x).$$

Secondly, assume that $x = v_i$ for some $i \in I_T$. Fix any two indices $j, k \in Inc_i$, $j \neq k$, and let φ be an upper (j, k) -test function of u at x . Setting $\varphi_\theta := \theta\varphi + (1 - \theta)\psi$ and observing that by definition 2.2 φ_θ is an upper (j, k) -test function of u_θ at x , we again obtain (6.6). Hence u_θ is a viscosity subsolution of

$$H(x, \partial_j u) \leq (1 - \theta) h^j(x).$$

Applying theorem 5.1 with $f = (1 - \theta)h$ and (6.5), it follows $u_\theta \leq v$ for all $\theta \in (0, 1)$. Letting θ tend to 1 yields $u \leq v$. Exchanging the role of u and v we conclude that $u = v$ in $\bar{\Gamma}$. \square

Remark 6.1. *For the problem $|Du|^2 - \alpha(x) = 0$ (see Remark 2.1) the existence of a strict subsolution follows by setting $K := \{x \in \Gamma : \alpha(x) = 0\}$, $\psi := C$ for some suitable $C \in \mathbb{R}$, and $h(x) := -\alpha(x)$.*

If g does not satisfy assumption (6.4) we can still characterize S as the maximal solution of the problem.

Proposition 6.2. *Let $g : \bar{\Gamma} \rightarrow \mathbb{R}$ be a continuous function. Then*

$$u(x) := \min\{g(y) + S(x, y) : y \in K \cup \partial\Gamma\}$$

is the maximal solution of (6.1) among the solutions v of (6.1) which satisfy $v \leq g$ on $K \cup \partial\Gamma$.

Proof. By Proposition 3.5, u is a solution of (2.9). If v is a solution of (6.1), then by (4.4)

$$v(x) \leq v(y) + S(y, x) \leq g(y) + S(y, x) \quad \text{for any } y \in K \cup \partial\Gamma,$$

and therefore the statement follows by Theorem 5.1. \square

Remark 6.2. *As explained in the introduction, a motivation of our work comes from the shortest path problem on a network. The case of a weighted graph studied in graph theory fits in our framework. In fact it is sufficient to choose the function α_j in Remark 2.1 in such a way that its integral along the edge e_j is equal to the given weight. We will study this problem in more details in a forthcoming paper.*

Acknowledgement 2. *This work is based on earlier results contained in the Ph.D. thesis of the first author. He would like to express his gratitude to Prof. K. P. Hadeler for the support and the guidance during the completion of the thesis.*

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